The bound state problem for a Dirac particle in an external static charge distribution in (1 + 1) dimensions

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Abstract
We study the self-interaction effects for the Dirac particle moving in an external field created by static charges in (1 + 1) dimensions. Assuming that the total electric charge of the system vanishes, we show that the asymptotically linearly rising part of the external potential responsible for the nonexistence of bound states in the external field problem without self-interaction is cancelled by the self-potential of the zero mode of the Dirac particle charge density. We derive the Dirac equation which includes the self-potential of the non-zero modes and is nonlinear. We solve the spectrum problem in the case of two external positive charges of the same value and prove that the Dirac particle and external charges are confined in a stable system.

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1. Introduction

The problem of motion of the Dirac particle in an external field can be studied in different approximations. In the vanishing self-interaction approximation we neglect the effects of self-interaction, i.e. the fact that the Dirac particle creates its own radiation field and interacts with it. Only the interaction between the Dirac particle and external field is taken into account. If the external field is created by static charges, then this interaction is described by the Coulomb potential.

However, the vanishing self-interaction approximation cannot be applied universally; in some cases it leads to incorrect results and even to paradoxes. In particular, if we take in (1+1) dimensions a system of two spin-$\frac{1}{2}$ particles of opposite charges coupled by an instantaneous Coulomb interaction and assume that one of the particles is much heavier (proton), while the lighter particle (electron) moves in the Coulomb field of the heavier one, then it turns out that the system does not have discrete energy levels [1]. In other words, in (1 + 1) dimensions

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the hydrogen atom does not exist. That happens not only for hydrogen-like systems with an infinitely heavy source of potential, but also for positronium-like systems.

It is therefore of principal importance to include self-interaction. We need the self-field effects to obtain the full picture of the interaction between the electromagnetic field and Dirac matter as well as to make that picture self-consistent. In the self-field formulation, the electromagnetic field has as its source all the charged particles which in turn move in this field. The total electromagnetic field is the sum of external and self-field parts. The external part is created by some external sources which are not dynamically relevant, and the self-field is created by the Dirac particle itself.

In this paper we aim to study the effects of self-interaction for the Dirac particle in $(1 + 1)$ dimensions in connection with the result of [1]. We want to determine whether in the presence of the self-field the Dirac particle and external charges can be confined in a stable system characterized by discrete energy levels.

Models in $(1 + 1)$ dimensions are interesting as simpler models for the discussion of different aspects of realistic particle physics models in $(3 + 1)$ dimensions. At the same time, $(1 + 1)$-dimensional models are interesting in their own right and have some peculiarities which make the physics in $(1 + 1)$ dimensions different (in principle) from the physics in $(3+1)$ dimensions. One of these peculiarities is that the Coulomb potential on the line is rising linearly at spatial infinities. Only the linear Coulomb potential is responsible for the paradox mentioned above.

One-dimensional models of spin-$\frac{1}{2}$ particles also have applications in condensed matter physics. It is enough to mention the one-dimensional electronic liquid model or the Thirring model. The problem of the existence of bound states for a Dirac particle in the presence of a static charge distribution may be useful for understanding the formation and decay of bound states for these particles in other one-dimensional models with more complicated interaction.

This paper is organized as follows. In section 2, we first neglect self-interaction and consider the Dirac particle moving in a potential created by external static charges. We find the asymptotics of the eigenfunctions of the Dirac Hamiltonian at spatial infinities. In accordance with [1], for any value of energy the eigenfunctions are not normalizable and cannot correspond to a discrete spectrum. In section 3, we introduce the self-field of the Dirac particle. Following the method of [2–4], we derive the Dirac equation which includes the nonlinear self-field term. In section 4, we study this equation for a specific choice of the external potential, namely for the case of two external positive charges of the same value. We solve the spectrum problem in the approximation when the discrete energy values are determined by the interaction between the Dirac particle and external field, while the self-interaction shifts these values by a small amount. Section 5 contains our conclusions.

2. The Dirac particle in an external field

The Dirac equation for a particle of charge e and mass m moving in an external field is

$$[\gamma^\mu(i\hbar c\partial_\mu - eA_\mu^{\text{ext}}) - mc^2]\psi(x) = 0 \tag{2.1}$$

where $(\mu, \nu = \overline{0, 1})$, γ^μ are 2×2 Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the field $\psi(x) = \psi(x_1, t)$ is the two-component Dirac spinor. The partial derivatives are defined as $\partial_0 = \partial/c\partial t$, $\partial_1 = \partial/\partial x_1$.

We assume that the external field is defined by

$$A_\mu^{\text{ext}}(x_1, t) = (A_0^{\text{ext}} = A_0^{\text{ext}}(x_1), A_1^{\text{ext}} = 0)$$

which is a time-independent scalar potential. We assume next that the potential is created by static charges, so we can use the following ansatz:

$$A_0^{\text{ext}}(x_1) = -\tfrac{1}{2} Q^{\text{ext}} |x_1| + \bar{A}_0^{\text{ext}}(x_1) \tag{2.2}$$

where Q^{ext} is total external charge, and $\bar{A}_0^{\text{ext}}$ is a potential which does not increase at spatial infinities, i.e. $\bar{A}_0^{\text{ext}}(x_1 = \pm\infty)$ are finite constants.

In particular, one external charge q at the point $x_1 = 0$ creates the current $j_0^{\text{ext}} = q\delta(x_1)$ that corresponds to the scalar potential $A_0^{\text{ext}} = -\tfrac{1}{2}q|x_1|$. Two external charges of the same q value taken at the points $x_1 = a$ and $-a$ create the current $j_0^{\text{ext}} = q(\delta(x_1 - a) + \delta(x_1 + a))$ and potential of the form (2.2) with $Q^{\text{ext}} = 2q$ and

$$\bar{A}_0^{\text{ext}} = \begin{cases} q(|x_1| - a) & \text{for} \quad |x_1| \leqslant a \\ 0 & \text{for} \quad |x_1| \geqslant a. \end{cases} \tag{2.3}$$

We can easily check that the ansatz (2.2) is valid for an arbitrary number of external static charges $q_1, q_2, q_3, \ldots$. Therefore, the potential for these charges can be always given as the sum of two parts, linearly rising and finite at spatial infinities.

If we act on both sides of (2.1) by $[-\gamma^\mu(i\hbar c\partial_\mu - eA_\mu^{\text{ext}}) - mc^2]$, then we come to the second-order differential equation for ψ [5,6]

$$\left[D_\mu D^\mu + eS^{\lambda\mu} F_{\lambda\mu}^{\text{ext}} + \frac{m^2 c^2}{\hbar^2} \right] \psi = 0 \tag{2.4}$$

where

$$D_\mu \equiv \partial_\mu + i\frac{e}{\hbar c} A_\mu^{\text{ext}}.$$

This equation looks like the Klein–Gordon one, except the additional term $eS^{\lambda\mu} F_{\lambda\mu}^{\text{ext}}$ with

$$S^{\lambda\mu} \equiv \tfrac{1}{2}i[\gamma^\lambda, \gamma^\mu]_- \qquad F_{\lambda\mu}^{\text{ext}} \equiv \partial_\lambda A_\mu^{\text{ext}} - \partial_\mu A_\lambda^{\text{ext}}.$$

In (3 + 1) dimensions, the spatial components of $S^{\lambda\mu}$ are related to the spin of the Dirac particle, so the additional term describes the interaction between the particle spin and the electromagnetic field.

In (1 + 1) dimensions, $S^{\lambda\mu}$ has no spatial components, and we cannot therefore introduce spin. The only nonvanishing component $S^{01} = i\alpha$, where $\alpha \equiv \gamma^5 = \gamma^0\gamma^1$, allows us to introduce chirality. If we write ψ in the component form as

$$\psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and define the operators

$$P_\pm \equiv \tfrac{1}{2}(1 \pm \alpha)$$

then

$$P_+\psi = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \qquad P_-\psi = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}$$

i.e. the upper component is of positive chirality and the lower one of negative chirality.

Moreover, there is no magnetic field in (1 + 1) dimensions, so the only nonvanishing component of $F_{\lambda\mu}^{\text{ext}}$ is

$$F_{01}^{\text{ext}} = -\frac{\partial A_0^{\text{ext}}}{\partial x_1} \equiv \mathcal{E}^{\text{ext}}$$

where $\mathcal{E}^{\text{ext}}$ is the external electric field. The additional term in (4) becomes

$$eS^{\lambda\mu}F_{\lambda\mu}^{\text{ext}} = \mathrm{i}e\alpha\mathcal{E}^{\text{ext}}$$

indicating that positive and negative chirality components are coupled differently to the external electric field.

In the Hamiltonian form equation (2.1) reads

$$H\psi \equiv \mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \left(\mathrm{i}\hbar c\alpha\frac{\partial}{\partial x_1} + \beta mc^2 + eA_0^{\text{ext}}\right)\psi \tag{2.5}$$

where $\beta \equiv \gamma^0$. The eigenvalue problem for the Dirac Hamiltonian, $H\psi = E\psi$, reduces to the problem of solving the system of two equations

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial x_1} + \frac{eA_0^{\text{ext}} - E}{c}\right)u_1 = -mcu_2 \tag{2.6a}$$

$$\left(-\mathrm{i}\hbar\frac{\partial}{\partial x_1} + \frac{eA_0^{\text{ext}} - E}{c}\right)u_2 = -mcu_1. \tag{2.6b}$$

We easily decouple u_1 and u_2 and rewrite these equations equivalently as

$$\left[\hbar^2\frac{\partial^2}{\partial x_1^2} \pm \mathrm{i}\frac{e\hbar}{c}\mathcal{E}^{\text{ext}} + \left(\frac{eA_0^{\text{ext}} - E}{c}\right)^2\right]u_{1(2)} = m^2c^2u_{1(2)} \tag{2.7}$$

the sign (+) corresponding to the positive chirality component and (−) to the negative one.

If the Dirac particle and external charges are confined in a stable system, then equations (2.7) must reveal a set of bound states. For a discrete set of energies in the band $|E| < mc^2$, these equations must have solutions which decrease exponentially at infinities and are normalizable.

However, for $Q^{\text{ext}} \neq 0$, the external potential is asymptotically linearly rising at spatial infinities, so the term $(e^2(Q^{\text{ext}})^2 x_1^2)/(4c^2)$ dominates in equations (2.7) and prevents any bound state. Indeed, the asymptotics of $u_{1(2)}$ for all possible energies in the band are determined by the equation

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{e^2(Q^{\text{ext}})^2}{4c^2\hbar^2}x_1^2\right]u_{1(2)}(|x_1| \to \infty) = 0. \tag{2.8}$$

This is the inverted oscillator equation [7]. Its general solution can be expressed in terms of parabolic cylinder functions. One way to choose two linearly independent solutions of (2.8) is to take the real functions $W(0, x_1)$ and $W(0, -x_1)$ (we follow the notations of [7]) whose asymptotic behaviour is well known:

$$W(0, x_1 \to +\infty) \approx \frac{1}{\sqrt{x_1}}\cos\left(\frac{|eQ^{\text{ext}}|}{c\hbar}x_1 + \frac{\pi}{4}\right)$$

$$W(0, x_1 \to -\infty) \approx \frac{1}{\sqrt{|x_1|}}\sin\left(\frac{|eQ^{\text{ext}}|}{c\hbar}x_1 + \frac{\pi}{4}\right).$$

We can represent $u_{1(2)}(|x_1| \to \infty)$ as linear combinations of $W(0, x_1 \to \pm\infty)$ with arbitrary coefficients. Regardless of the choice for these coefficients, the normalization integral

$$\int^{x_1} \mathrm{d}x_1' \left(|u_1(x_1')|^2 + |u_2(x_1')|^2\right)$$

diverges for large x_1. This means that the probability of finding the Dirac particle and external charges infinitely separated remains finite at all times. Consequently, equations (2.7) have solutions for an arbitrary energy, but these solutions can never represent a bound state.

3. Self-interaction

Let us now introduce the self-field of the Dirac particle and investigate how it influences the bound state spectrum. To derive an equation for the Dirac particle in the presence of both the external and its own radiation fields, we start first with the action

$$S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx_1 \, [\bar{\psi}(x_1, t)(\gamma^\mu i\hbar c \partial_\mu - mc^2)\psi(x_1, t) - j_\mu^{tot}(x_1, t) A^\mu(x_1, t)$$
$$- \tfrac{1}{4} F_{\lambda\mu}(x_1, t) F^{\lambda\mu}(x_1, t)] \tag{3.1}$$

where

$$j_\mu^{tot} = j_\mu + j_\mu^{ext}$$

is a total current which includes both the Dirac matter current $j^\mu \equiv e\bar{\psi}\gamma^\mu\psi$ and the one created by external static charges.

If we expand the field ψ into the Fourier integral

$$\psi(x_1, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \cdot e^{\frac{i}{\hbar} p x_1} \psi(p, t)$$

then the charge density of the Dirac matter becomes

$$j_0(x_1, t) = e\psi^\star(x_1, t)\psi(x_1, t) = \frac{e}{2\pi} \int_{-\infty}^{\infty} dk \cdot e^{-\frac{i}{\hbar} k x_1} \rho(k, t) \tag{3.2}$$

where

$$\rho(k, t) \equiv \int_{-\infty}^{\infty} dp \, \psi^\star(p + k, t)\psi(p, t).$$

The zero-momentum component of the density determines the matter charge

$$Q \equiv \int_{-\infty}^{\infty} dx \, j_0(x_1, t) = e\hbar\rho(0, t).$$

Separating in (3.2) $\rho(0, t)$ and $\rho(k, t)$ with the non-zero momentums $k \neq 0$, we rewrite (3.2) as

$$j_0(x_1, t) = Q\delta(x_1) + \bar{j}_0(x_1, t) \tag{3.3}$$

the density

$$\bar{j}_0(x_1, t) \equiv \frac{e}{2\pi} \int_{-\infty}^{\infty} dk \cdot e^{-\frac{i}{\hbar} k x_1} \bar{\rho}(k, t)$$

$$\bar{\rho}(k, t) \equiv \rho(k, t) - \rho(0, t)$$

corresponding to a zero charge, $\int_{-\infty}^{\infty} dx_1 \, \bar{j}_0(x_1, t) = 0$.

With the choice of the gauge $\partial_\mu A^\mu = 0$, the Maxwell equations take the form

$$\Box A_\mu = j_\mu^{tot} \tag{3.4}$$

and are solved by

$$A_\mu(x_1, t) = c \int_{-\infty}^{\infty} dx_1' \int_{-\infty}^{\infty} dt' \, D^c(x_1 - x_1'; t - t') j_\mu^{tot}(x_1', t') = A_\mu^{self}(x_1, t) + A_\mu^{ext}(x_1, t) \tag{3.5}$$

where $D^c(x_1, t)$ is the causal Green function

$$\Box D^c(x_1, t) = \delta(x_1)\delta(ct)$$
$$D^c(x_1, t) \equiv -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq_1 \frac{1}{q_0^2 - q_1^2 + i\epsilon} e^{-\frac{i}{\hbar} c q_0 t} e^{\frac{i}{\hbar} q_1 x_1} \tag{3.6}$$

and $\Box \equiv \partial_\mu \partial^\mu$.

Substituting (3.3) and (3.6) into the expression for the electromagnetic field, we obtain its external and self-field parts in the form

$$A_\mu^{\text{ext}}(x_1, t) = (-\tfrac{1}{2} Q^{\text{ext}} |x_1| + \bar{A}_0^{\text{ext}}(x_1)) \delta_{\mu 0}$$

that agrees with (2.2), and

$$A_\mu^{\text{self}}(x_1, t) = -\tfrac{1}{2} Q |x_1| \delta_{\mu 0} + \bar{A}_\mu^{\text{self}}(x_1, t)$$

with

$$\bar{A}_\mu^{\text{self}}(x_1, t) \equiv c \int_{-\infty}^{\infty} \mathrm{d}x_1' \int_{-\infty}^{\infty} \mathrm{d}t' \, D^c(x_1 - x_1'; t - t') \bar{j}_\mu(x_1', t')$$

and

$$\bar{j}_\mu(x_1, t) \equiv j_\mu(x_1, t) - Q \delta_{\mu 0} \delta(x_1).$$

Putting both parts together, we see that if the total charge of the system vanishes

$$Q^{\text{tot}} \equiv Q + Q^{\text{ext}} = 0 \tag{3.7}$$

then the linearly rising potentials produced by the Dirac particle and external charges cancel each other, and for the total electromagnetic field we get

$$A_\mu(x_1, t) = \bar{A}_\mu^{\text{self}}(x_1, t) + \bar{A}_0^{\text{ext}}(x_1) \delta_{\mu 0}. \tag{3.8}$$

One of the Maxwell equations (3.4) is the Gauss' law

$$\frac{\partial}{\partial x_1} \mathcal{E} = j_0^{\text{tot}} \tag{3.9}$$

where the electric field also consists of two parts, self-field and external,

$$\mathcal{E} = F_{01} = \mathcal{E}^{\text{self}} + \mathcal{E}^{\text{ext}}.$$

Integrating equation (3.9) over x_1, we obtain

$$\mathcal{E}^{\text{self}}(+\infty) - \mathcal{E}^{\text{self}}(-\infty) = -(\mathcal{E}^{\text{ext}}(+\infty) - \mathcal{E}^{\text{ext}}(-\infty))$$

i.e. the difference in the values of the self-field electric field at the ends of x_1-line balances the one of the external electric field, and for the total electric field $\mathcal{E}(+\infty) = \mathcal{E}(-\infty)$.

The vanishing of the total charge therefore allows one to balance the sources of electric flux and is important from a physical point of view. For $Q^{\text{tot}} \neq 0$, the balance is destroyed, and, as we have seen in the previous section, the Dirac particle and external charges escape from each other and cannot create a stable system.

The condition (3.7) has its analogues in the second-quantized version of different $(1 + 1)$-dimensional models. In the Schwinger model [8], the total electric charge is known to be zero on the physical states [9, 10]. According to Schwinger [8], when a charge is inserted into the vacuum, the accompanying electric field polarizes the vacuum producing complete compensation of the charge. In $(1 + 1)$-dimensional QCD, we restrict ourselves to colour-neutral states, since the presence of uncompensated colour charge in space leads to a growth of the fields at infinities and makes the total energy of the system infinite [11].

Inserting (3.8) into the action and using a partial integration, we have

$$S = S_0 + S_{\text{self}}$$

$$S_0 \equiv \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x_1 \, [\bar{\psi}(x_1, t)(\gamma^\mu \mathrm{i}\hbar c \partial_\mu - mc^2)\psi(x_1, t)$$
$$\qquad - e\bar{\psi}(x_1, t)\gamma^\mu \psi(x_1, t) \cdot \bar{A}_\mu^{\text{ext}} - \tfrac{1}{2} j_{\text{ext}}^\mu \bar{A}_\mu^{\text{ext}}] \tag{3.10}$$

$$S_{\text{self}} \equiv \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x_1 \, [-e\bar{\psi}(x_1, t)\gamma^\mu \psi(x_1, t) \bar{A}_\mu^{\text{self}} - \tfrac{1}{4} \bar{F}_{\lambda\mu}^{\text{self}} \bar{F}_{\text{self}}^{\lambda\mu}]$$
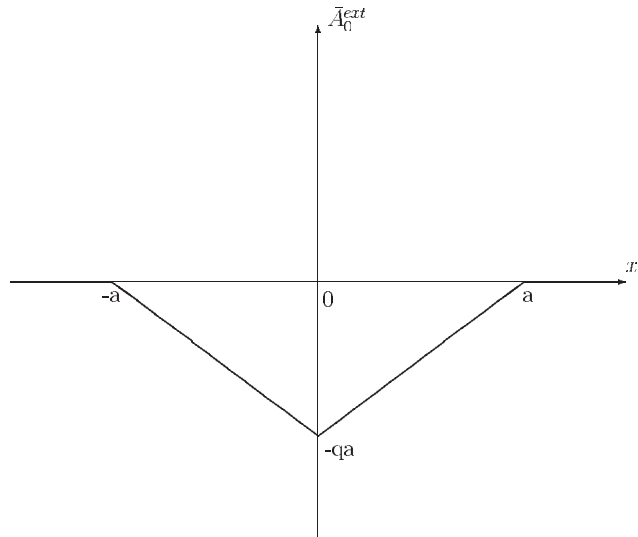
Figure 1. The potential $\bar{A}_0(x_1)$ in the case of two external charges of the same value $q > 0$ at the points $x_1 = \pm a$.

with $\bar{F}^{\text{self}}_{\lambda\mu} \equiv \partial_\lambda \bar{A}^{\text{self}}_\mu - \partial_\mu \bar{A}^{\text{self}}_\lambda$ and $\bar{A}^{\text{ext}}_\mu \equiv \bar{A}^{\text{ext}}_0 \delta_{\mu 0}$. Variation of this action with respect to the Dirac field yields the following Dirac equation:

$$[\gamma^\mu(i\hbar c\partial_\mu - e\bar{A}^{\text{ext}}_\mu) - mc^2]\psi(x) = e\gamma^\mu \bar{A}^{\text{self}}_\mu \psi(x). \tag{3.11}$$

It is essential that neither the linearly rising part of the external electric field nor the one of the self-field enter this equation.

Since $\bar{A}^{\text{self}}_\mu$ is expressed in terms of the current $\bar{j}_\mu$, (3.11) is a nonlinear integral equation for ψ. In the next section, we will continue our study of equation (3.11) for a specific choice of the external electric field.

4. Example

(1) As an example, let us consider in detail the case of two external charges of the same value $q > 0$ at the points $x_1 = \pm a$ and find first the spectrum for the vanishing $\bar{A}^{\text{self}}_\mu$.

With the external potential $\bar{A}^{\text{ext}}_0$ given by (2.3) (see figure 1), the Dirac Hamiltonian is invariant under a modified parity transformation generated by

$$\hat{T} \equiv \beta \hat{I} \tag{4.1}$$

where the action of $\hat{I}$ on any function of x_1 is defined as

$$\hat{I} f(x_1) = f(-x_1).$$

The Hamiltonian eigenfunctions must therefore be even or odd under parity reversal. Since the transformation for ψ is

$$\psi(x_1, t) \rightarrow \psi'(x_1, t) = \beta\psi(-x_1, t)$$

we get

$$u_1(x_1, t) = u_2(-x_1, t)$$

for even eigenfunctions, and

$$u_1(x_1, t) = -u_2(-x_1, t)$$

for odd ones.

To solve the system (2.6a), (2.6b) with $\bar{A}_0^{\text{ext}}$, we have to find general solutions in the regions $-a \leqslant x_1 \leqslant a$, $x_1 \geqslant a$ and $x_1 \leqslant -a$, and match them at the boundaries so that the eigenfunction and its first derivative is continuous. We also wish to distinguish the regions of positive and negative values of x_1, and so introduce

$$u_{1(2)}^+ \equiv u_{1(2)}(x_1 > 0) \qquad u_{1(2)}^- \equiv u_{1(2)}(x_1 < 0).$$

To simplify the notation, we define in the region $0 \leqslant x_1 \leqslant a$ the new dimensionless variable

$$z_+ \equiv \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \sqrt{\frac{eq}{\hbar c}} \left(x_1 - a - \frac{E}{eq} \right). \tag{4.2}$$

The system (2.6a), (2.6b) then becomes

$$\left(\frac{\partial}{\partial z_+} + z_+ \right) u_1^+ = \mathrm{i}\sqrt{\frac{2}{\Delta}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} u_2^+ \tag{4.3a}$$

$$\left(-\frac{\partial}{\partial z_+} + z_+ \right) u_2^+ = \mathrm{i}\sqrt{\frac{2}{\Delta}} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} u_1^+ \tag{4.3b}$$

where $\Delta \equiv (2eq\hbar)/(m^2 c^3)$ is a dimensionless constant. Decoupling u_1^+ and u_2^+, we obtain the system of two second-order differential equations

$$\frac{\partial^2}{\partial z_+^2} u_{1(2)}^+ + [(2\nu \pm 1) - z_+^2] u_{1(2)}^+ = 0 \tag{4.4}$$

with $\nu \equiv -\mathrm{i}/\Delta$.

The system (4.4) is solved by hypergeometric functions. If we introduce

$$y_+ \equiv z_+^2 \qquad v_{1(2)}^+ \equiv \mathrm{e}^{\frac{1}{2}z_+^2} \cdot u_{1(2)}^+$$

then (4.4) reduces to

$$y_+ \frac{\partial^2}{\partial y_+^2} v_{1,+} + \left(\frac{1}{2} - y_+ \right) \frac{\partial}{\partial y_+} v_{1,+} + \frac{\nu}{2} v_{1,+} = 0 \tag{4.5a}$$

$$y_+ \frac{\partial^2}{\partial y_+^2} v_{2,+} + \left(\frac{1}{2} - y_+ \right) \frac{\partial}{\partial y_+} v_{2,+} + \frac{\nu - 1}{2} v_{2,+} = 0 \tag{4.5b}$$

which are just the hypergeometric function equations. For the first equation, the linearly independent solutions are $F(-\frac{\nu}{2}; \frac{1}{2}; y_+)$ and $y_+^{1/2} F(\frac{1-\nu}{2}; \frac{3}{2}; y_+)$, while for the second one $F(\frac{1-\nu}{2}; \frac{1}{2}; y_+)$ and $y_+^{1/2} F(1 - \frac{\nu}{2}; \frac{3}{2}; y_+)$.

A similar analysis can be performed in the region $-a \leqslant x_1 \leqslant 0$. With the variable

$$z_- \equiv \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \sqrt{\frac{eq}{\hbar c}} \left(x_1 + a + \frac{E}{eq} \right)$$

the equations for $u_{1(2)}^-$ read

$$\frac{\partial^2}{\partial z_-^2} u_{1(2)}^- + [(2\nu \mp 1) - z_-^2] u_{1(2)}^- = 0. \tag{4.6}$$

The linearly independent solutions for $v_{1(2)}^- \equiv \mathrm{e}^{\frac{1}{2}z_-^2} \cdot u_{1(2)}^-$ are $F(\frac{1-\nu}{2}; \frac{1}{2}; y_-)$, $y_-^{1/2} F(1 - \frac{\nu}{2}; \frac{3}{2}; y_-)$ and $F(-\frac{\nu}{2}; \frac{1}{2}; y_-)$, $y_-^{1/2} F(\frac{1-\nu}{2}; \frac{3}{2}; y_-)$, correspondingly, where $y_- \equiv z_-^2$.

Taking linear combinations of these solutions, we can construct eigenfunctions

$$\psi^\pm \equiv \begin{pmatrix} u_1^\pm \\ u_2^\pm \end{pmatrix}$$

which fulfil the matching conditions

$$\psi^\pm(x_1 = +0) = \psi^-(x_1 = -0) \qquad \frac{\partial \psi^+}{\partial x_1}(x_1 = +0) = \frac{\partial \psi^-}{\partial x_1}(x_1 = -0)$$

and are even or odd under parity reversal.

The even eigenfunctions up to a constant factor are

$$\psi_{\text{even}}^+ = e^{-\frac{1}{2}z_+^2} \begin{pmatrix} \sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(-\frac{\nu}{2}; \frac{1}{2}; z_+^2) + G z_+ F(\frac{1-\nu}{2}; \frac{3}{2}; z_+^2) \\ z_+ F(1 - \frac{\nu}{2}; \frac{3}{2}; z_+^2) - G\sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(\frac{1-\nu}{2}; \frac{1}{2}; z_+^2) \end{pmatrix} \tag{4.7a}$$

for $0 \leqslant x_1 \leqslant a$, and

$$\psi_{\text{even}}^- = e^{-\frac{1}{2}z_-^2} \begin{pmatrix} -z_- F(1 - \frac{\nu}{2}; \frac{3}{2}; z_-^2) - G\sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(\frac{1-\nu}{2}; \frac{1}{2}; z_-^2) \\ \sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(-\frac{\nu}{2}; \frac{1}{2}; z_-^2) - G z_- F(\frac{1-\nu}{2}; \frac{3}{2}; z_-^2) \end{pmatrix} \tag{4.7b}$$

for $-a \leqslant x_1 \leqslant 0$. The constant

$$G = G(\Delta, z_a) \equiv \frac{2\nu z_a F(1 - \frac{\nu}{2}; \frac{3}{2}; 2\nu z_a^2) + F(-\frac{\nu}{2}; \frac{1}{2}; 2\nu z_a^2)}{2\nu z_a F(\frac{1-\nu}{2}; \frac{3}{2}; 2\nu z_a^2) - F(\frac{1-\nu}{2}; \frac{1}{2}; 2\nu z_a^2)}$$

is modulo 1, $|G|^2 = 1$, and $z_a \equiv (E + eqa)/(mc^2)$.

The odd eigenfunctions are

$$\psi_{\text{odd}}^+ = e^{-\frac{1}{2}z_+^2} \begin{pmatrix} \sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(-\frac{\nu}{2}; \frac{1}{2}; z_+^2) + \bar{G} z_+ F(\frac{1-\nu}{2}; \frac{3}{2}; z_+^2) \\ z_+ F(1 - \frac{\nu}{2}; \frac{3}{2}; z_+^2) - \bar{G}\sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(\frac{1-\nu}{2}; \frac{3}{2}; z_+^2) \end{pmatrix} \tag{4.8a}$$

for $0 \leqslant x_1 \leqslant a$, and

$$\psi_{\text{odd}}^- = e^{-\frac{1}{2}z_-^2} \begin{pmatrix} z_- F(1 - \frac{\nu}{2}; \frac{3}{2}; z_-^2) + \bar{G}\sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(\frac{1-\nu}{2}; \frac{1}{2}; z_-^2) \\ -\sqrt{\frac{\Delta}{2}} e^{i\frac{\pi}{4}} F(-\frac{\nu}{2}; \frac{1}{2}; z_-^2) + \bar{G} z_- F(\frac{1-\nu}{2}; \frac{3}{2}; z_-^2) \end{pmatrix} \tag{4.8b}$$

for $-a \leqslant x_1 \leqslant 0$, where

$$\bar{G}(\Delta, z_a) = -G(\Delta, -z_a).$$

For $x_1 \geqslant a$ it can be checked that

$$\psi_{\text{even}}^+ = s \begin{pmatrix} E - i\kappa \\ mc^2 \end{pmatrix} e^{-\frac{1}{\hbar c}\kappa x_1} \tag{4.9}$$

with

$$\kappa \equiv \sqrt{m^2 c^4 - E^2}$$

satisfies the Dirac equation. Matching (4.7a) and (4.9) at $x_1 = a$ and eliminating s we obtain the equation that determines the spectrum of the even bound states

$$E = mc^2 \cos(\lambda_G(E)) \tag{4.10}$$

where

$$\lambda_G \equiv \arg \left(\frac{2\nu G z_0 F(\frac{1-\nu}{2}; \frac{3}{2}; 2\nu z_0^2) - F(-\frac{\nu}{2}; \frac{1}{2}; 2\nu z_0^2)}{2\nu z_0 F(1 - \frac{\nu}{2}; \frac{3}{2}; 2\nu z_0^2) + G F(\frac{1-\nu}{2}; \frac{1}{2}; 2\nu z_0^2)} \right)$$

and $z_0 \equiv E/(mc^2)$. The matching condition at $x_1 = -a$ for (4.7b) and

$$\psi_{\text{even}}^- = s \begin{pmatrix} mc^2 \\ E - \mathrm{i}\kappa \end{pmatrix} \mathrm{e}^{\frac{1}{\hbar c}\kappa x_1} \qquad\qquad x_1 \leqslant -a \tag{4.11}$$

gives the same spectrum equation.

In a similar way we can derive the equation that determines the spectrum of the odd bound states:

$$E = mc^2 \cos(\lambda_{\tilde{G}}(E)). \tag{4.12}$$

(2) With the self-field term $\bar{A}_\mu^{\text{self}}$ the Dirac equation is nonlinear in ψ, so the spectrum problem becomes essentially more complicated. As in [2–4], we can solve the problem in the approximation when the self-interaction contribution to the energy spectrum is very small with respect to the contribution of the interaction between the Dirac particle and the external field.

Let us assume that equations (4.10) and (4.12) have N_+ and N_- solutions, correspondingly, i.e. for $\bar{A}_\mu^{\text{self}} = 0$ there are N_+ even and N_- odd bound states. The total number of discrete states in the band $|E| < mc^2$ is then $N = N_+ + N_-$. Let us denote the normalized bound state eigenfunctions by $\psi_n^{\text{ext}}, n = \overline{1, N}$. Up to a normalization factor ψ_n^{ext} coincide with (4.7a), (4.7b) for even states and with (4.8a), (4.8b) for odd states in the region $-a \leqslant x_1 \leqslant a$.

In $(3 + 1)$ dimensions, the self-field effects are of the order of the fine structure constant and higher. This allows us to restrict our calculations to the first order of this constant. In our study, to make the self-field effects small we assume that $|e| \ll q$, so the self-interaction shifts the bound state energies by a small amount

$$E_n = E_n^{\text{ext}} + \Delta E_n^{\text{self}} \tag{4.13}$$

and does not change the number of states. In (4.13) E_n^{ext} are discrete spectrum energies of the external field problem without $\bar{A}_\mu^{\text{self}}$. The eigenfunctions of the discrete spectrum cannot be characterized now by a definite parity, because the self-interaction term in the Dirac Hamiltonian is not in general invariant under parity reversal.

For the nonlinear Dirac equation the superposition principle does not hold. It is not possible to expand the exact solutions of this nonlinear equation as a superposition of the linear equation solutions ψ_n^{ext} with arbitrary time-dependent coefficients, even though ψ_n^{ext} form a complete set. What we can use now is the Fourier expansion in the time coordinate [2–4]

$$\psi(x_1, t) = \sum_{n=1}^{N} \psi_n(x_1)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t} + \int_{-\infty}^{-mc^2} \mathrm{d}E\, \psi(x_1, E)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Et} + \int_{mc^2}^{\infty} \mathrm{d}E\, \psi(x_1, E)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Et} \tag{4.14}$$

in which the time behaviour is known, and of the form $\exp(-\frac{\mathrm{i}}{\hbar}Et)$. The functions $\psi_n(x_1)$, $\psi(x_1, E)$ and the energies E_n are unknown.

To derive information about the spectrum it is simpler and more convenient to work with the action rather than with the Dirac equation. Since manipulations which we are going to do below are valid in both discrete and continuous spectrums, instead of (4.14) we can use the following compact expression:

$$\psi(x_1, t) = \overline{\sum}_n \psi_n(x_1)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t} \tag{4.15}$$

where $\overline{\sum}_n$ means summation over discrete states and integration over continuous ones.

If we insert the Fourier expansion into the action, then up to the terms depending only on the external field we obtain

$$S_0 = \overline{\sum}_{n,m} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x_1\, \bar{\psi}_n(x_1)[\gamma^\mu(\mathrm{i}\hbar c\partial_\mu - e\bar{A}_\mu^{\text{ext}}) - mc^2]\psi_m(x_1)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\omega_{nm}t}$$

where $\omega_{nm} \equiv E_n - E_m$, and

$$
\begin{aligned}
S_{\text{self}} = -\frac{e^2 c}{2} \overline{\sum}_{n,m,r,s} &\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx_1' \, [j_{nm}^\mu(x_1) D^c(x_1 - x_1'; t - t') j_{rs,\mu}(x_1') \\
&- j_{nm}^0(x_1) D^c(0; t - t') j_{rs,0}(x_1')] e^{\frac{i}{\hbar}(\omega_{nm}t + \omega_{rs}t')}
\end{aligned}
$$

where

$$
j_{nm}^\mu(x_1) \equiv \bar{\psi}_n(x_1) \gamma^\mu \psi_m(x_1).
$$

After time integration for S_0 we find

$$
S_0 = 2\pi\hbar \overline{\sum}_{n,m} \int_{-\infty}^{\infty} dx_1 \, \psi_n^\star(x_1)(E_m - H)\psi_m(x_1)\delta(\omega_{nm}) \tag{4.16}
$$

with H given by (2.5). If the $\psi_n(x_1)$ were solutions of the external field problem with the vanishing self-potential $\bar{A}_\mu^{\text{self}}$, i.e. $\psi_n^{\text{ext}}(x_1)$, then this expression would be zero for $E_n = E_n^{\text{ext}}$. The $\psi_n^{\text{ext}}(x_1)$ minimize S_0 alone. However, now the entire action S, of which S_0 is only one term, must be minimized as a whole.

Time integrations in S_{self} can be performed using (3.6) and we can write the self-field part of the action entirely in terms of the Fourier components of the currents:

$$
\begin{aligned}
S_{\text{self}} = \frac{e^2\hbar^2}{2} \overline{\sum}_{n,m,r,s} \delta(\omega_{nm} + \omega_{rs}) \int_{-\infty}^{\infty} &\frac{dq_1}{\frac{1}{c}\omega_{nm}^2 - q_1^2 + i\epsilon} \\
&\times (T_{nm}^\mu(q_1) T_{rs,\mu}(-q_1) - T_{nm}^0(0) T_{rs}^0(0))
\end{aligned} \tag{4.17}
$$

where

$$
T_{nm}^\mu(q_1) \equiv \int_{-\infty}^{\infty} dx_1 \, j_{nm}^\mu(x_1) e^{\frac{i}{\hbar}q_1 x_1}. \tag{4.18}
$$

The $\psi_n(x_1)$ here are still exact solutions for which the action S will vanish identically.

In the approximation of the small self-interaction contribution, we can solve the spectrum problem iteratively. To the lowest order of iteration we replace $\psi_n(x_1)$ with $\psi_n^{\text{ext}}(x_1)$ and then solve for E_n which have the form (4.13). The number of discrete states without and with the self-field term $\bar{A}_\mu^{\text{self}}$ is assumed to be the same, and transitions between discrete and continuous states are neglected. Omitting the integration over continuous states and using the orthonormality of ψ_n^{ext}, we write S_0 in the form

$$
S_0 = 2\pi\hbar \sum_{n,m=1}^{N} \Delta E_n^{\text{self}} \cdot \delta(\omega_{nm})\delta_{nm}. \tag{4.19}
$$

In the self-field part of the action we separate the terms according to $E_n = E_m$, $E_r = E_s$ and according to $E_s = E_n$, $E_r = E_m$, the only two ways of satisfying the overall δ-function for discrete spectrum energies. And since $S = 0$ to this order of iteration we can solve for ΔE_n^{self}. The action and the total energy of the system are related by a δ-function. Cancelling this δ-function as well as the sum over n to obtain the energy shift of a fixed level n, we get

$$
\begin{aligned}
\Delta E_n^{\text{self}} = \frac{e^2\hbar}{4\pi} \sum_{m=1}^{N} \int_{-\infty}^{\infty} dq_1 \, \mathcal{P}\frac{1}{q_1^2} \cdot G_{nn,mm}(q_1) &- \frac{e^2\hbar}{8\pi} \sum_{m=1}^{N} \\
\times \int_{-\infty}^{\infty} \frac{dq_1}{|q_1|} \mathcal{P} \frac{1}{\frac{1}{c}\omega_{nm} - |q_1|} &(G_{nm,mn}(q_1) + G_{mn,nm}(q_1)) + i\frac{e^2\hbar c}{4} \\
\times \sum_{(m<n)} \frac{1}{\omega_{nm}} \text{Re} &\left[G_{nm,mn}\left(\frac{1}{c}\omega_{nm}\right) + G_{nm,mn}\left(\frac{1}{c}\omega_{mn}\right) \right]
\end{aligned} \tag{4.20}
$$

where

$$G_{nm,rs}(q_1) \equiv T_{nm}^{\text{ext},\mu}(q_1)T_{rs,\mu}^{\text{ext}}(-q_1) - T_{nm}^{\text{ext},0}(0)T_{rs}^{\text{ext},0}(0)$$

$\mathcal{P}$ stands for the principal value prescription, while Re means the real part. The superscript ext here indicates that the Fourier components (4.18) must be calculated by using ψ_n^{ext}.

The last term in (4.20) contributes if $m < n$. This shows that only the ground state $n = 1$ of our N-level system is stable. The energy shift for the excited states $n = 2, \ldots, N$ is complex, so the imaginary part of the shift can be identified with the spontaneous emission of the excited states due to self-interaction.

5. Conclusions

(1) We have studied the bound state problem for the Dirac particle moving in both an external and its own radiation fields in $(1 + 1)$ dimensions. We have shown that if the total electric charge of the system vanishes, then the asymptotically linearly rising part of the external potential which was responsible for nonexistence of bound states in the external field problem without self-interaction is cancelled by the self-potential of the zero mode of the Dirac particle charge density. The resulting Dirac equation includes only that part of the external potential which is finite at spatial infinities and also the self-potential created by the non-zero modes of the charge density.

We have proved that this equation has a set of solutions which show that the Dirac particle and external charges are confined in a stable system. These solutions correspond to energy levels in the band $|E| < mc^2$. Only the lowest level is precise. The higher levels have a nonzero linewidth which manifests itself as spontaneous emission. This picture is characteristic for hydrogen-like atoms. According to the self-field approach, the hydrogen atom has no precisely defined sharp energy levels, other than the ground state [2–4]. The excited states cannot be stable due to radiation reaction.

(2) We have solved the bound state problem for the external potential created by two positive static charges of the same value. All other cases when we have two or more external charges can be considered analogously.

The exceptional case is one external charge at the origin. The potential created by this charge consists only of the asymptotically linearly rising part, so the corresponding nonlinear Dirac equation does not include any external field. This means that the approximation when the self-interaction contribution to the spectrum is small with respect to the one of the external field cannot be applied. We need here to look for other ways to solve the nonlinear Dirac equation. This problem remains open.

(3) Vanishing of the total electric charge is the condition necessary for the existence of bound states of the Dirac particle and external static charges on line. If the total electric charge is not zero, then the Dirac equation does not reveal bound state solutions. Therefore, in the Dirac theory on line bound states can be only neutral. This is one of the peculiarities of one-dimensional physics.

We can consider charged states as well, but these states are not stable. The presence of an uncompensated electric charge leads to the linearly rising Coulomb potential in the Dirac equation and to a potential of the inverted oscillator type in the corresponding Schrödinger equation. Such potentials are known to allow only metastable states [12]. So for limited times the Dirac particles and external charges can be confined in a charged metastable state. Metastable states can decay into stable ones. If particles emitted in decay take away a charge equal to the charge of the metastable state, then the remaining particles can create a neutral bound state.

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